Discrete Mathematics 2025 Spring

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CAMEA 中国高质量MBA教育认证
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- 5.1 Function Definition and Properties
- 5.2 Composition of Functions and Inverse Functions

5.1.1 Definition of a Function

- Equality of functions
- Function's domain, range, and direction
- Surjective, Injective, and Bijective Functions
- Function Sets and Counting
- Constant, Identity, and Monotonic Functions
- Natural mapping
- Equivalence relation’s impact on natural mapping
- The order of a complexity function

5.1.2 Image and Preimage of a Function

5.1.3 Properties of a Function

- **Definition 5.1:** *function*

  Let ***f*** be a binary relation. If for every ∀***x***∈**dom*f*** there exists a unique ***y***∈**ran*f*** such that ***x f y*** holds, then ***f*** is called a ***function***. For a function ***f***, if ***x f y***, we denote this as ***y*=*f*(*x*)**, and ***y*** *i*s called the value of ***f at x***.

- **For example:**

  $$f_1=\{<x_1,y_1>,<x_2,y_2>,<x_3,y_2>\}$$

  $$f_2=\{<x_1,y_1>,<x_1,y_2>\}$$

f_1 is a function, but f_2 is not a function.

- **Definition 5.2:** Equality of functions
  Let f, g be functions. Then, f, g are equal if and only if their set representations are equal: $f=g \Leftrightarrow f\subseteq g \wedge g\subseteq f$

- If two functions f *and* g are equal, the following two conditions must be satisfied:
  (1) dom$f$ = dom$g$
  (2) $\forall x \in$ dom$f$ = dom$g$
  we have $f(x) = g(x)$

- **Example:** $f(x)=(x^2-1)/(x+1)$, $g(x)=x-1$
  These functions are not equal because **dom$f \subset$dom$g$.**

- **Definition 5.3:** function $f{:}A \to B$

  Let A and B be sets. If

  (1) f is a function,

  (2) **dom*f* = *A and***

  (3) **ran*f* $\subseteq$ *B*,**

  then f is called a **function from *A to B***, denoted by $f : A \to B$.

- **Examples:**

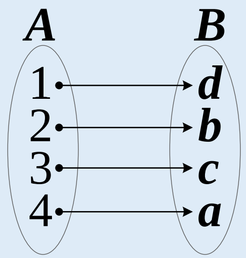
  $f : \mathrm{N} \to \mathrm{N}$, $f(x)=2x$ is a function from **N *to* N**

  $g : \mathrm{N} \to \mathrm{N}$, $g(x)=2$ is also a function from **N to N**

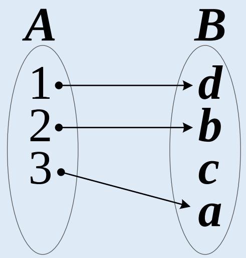
5.1.1 Definition of a Function
↳ Surjective, Injective, and Bijective Functions

- **Definition 5.4:** Let $f : A \to B$
  - A function $f: A \to B$ is ***surjective (onto)*** if and only if for every $b \in B$, there exists an $a \in A$ such that $f(a)=b$.
  - Function $f: A \to B$ is ***injective*** if and only if for all $a, b \in A$, we have $f(a)= f(b) \Rightarrow a=b$.
  - If $f:A \to B$ is both **surjective** and **injective**, then it is called a ***bijective*** function (or **bijection**).

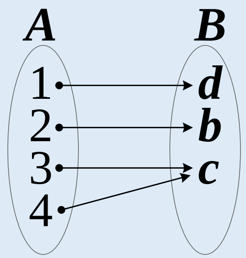
- **Examples:**



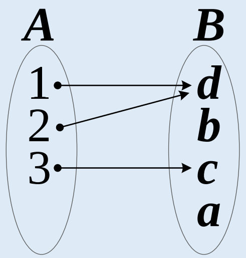
Bijective (Injective and Surjective)



Injective but not Surjective



Surjective but not Injective



Neither Surjective nor Injective

- **Definition 5.5:** The set of all functions from ***A to B*** is denoted by B^A, **read as** "B to the power of A"

  In symbolic form:

$$B^A = \{ f \mid f : A \to B \}$$

- **Counting B^A :**
  - $|A|=m$, $|B|=n$, and $m, n>0$, $|B^A|=n^m$.
  - $A=\varnothing$, then $B^A=B^{\varnothing}=\{\varnothing\}$.

    The function set contains only one element: the empty function. $|B^{\varnothing}|=1$
  - $A\neq\varnothing$ and $B=\varnothing$, then $B^A=\varnothing^A=\varnothing$.

    here is no function from a non-empty set to the empty set. $|\varnothing^A|=0$

- **Example: Let $A = \{1, 2, 3\}$, $B = \{a, b\}$, solve B^A.**

Solve: Find all possible functions from A to B.

$B^A = \{ f_0, f_1, \ldots , f_7 \}$, then

$f_0=\{<1,a>,<2,a>,<3,a>\}$

$f_1=\{<1,a>,<2,a>,<3,b>\}$

$f_2=\{<1,a>,<2,b>,<3,a>\}$

$f_3=\{<1,a>,<2,b>,<3,b>\}$

$f_4=\{<1,b>,<2,a>,<3,a>\}$

$f_5=\{<1,b>,<2,a>,<3,b>\}$

$f_6=\{<1,b>,<2,b>,<3,a>\}$

$f_7=\{<1,b>,<2,b>,<3,b>\}$

Definition 5.6:

(1) Let f: $A \to B$, If there exists a constant $c \in B$ such that for all $x \in A$, $f(x)=c$, then $f: A \to B$ is called a ***constant function***.

(2) The **identity relation** I_A on A is called the ***identity function*** on A, where for all $x \in A$, $I_A(x)=x$.

(3) Let $<A, \preccurlyeq>$, $<B, \preccurlyeq>$ be partially ordered sets, $f: A \to B$ called ***monotonically increasing*** (or simply **monotonic**) if for any x_1, $x_2 \in A$, $x_1 \prec x_2$, $\Rightarrow f(x_1) \preccurlyeq f(x_2)$;

- ***strictly monotonically increasing*** if for any x_1, $x_2 \in A$, $x_1 \prec x_2$, , $\Rightarrow f(x_1) \prec f(x_2)$.
- Similarly, **monotonically decreasing** and **strictly monotonically decreasing** functions can be defined in the same manner.

(4) Let *A* be a set. For any subset $A' \subseteq A$, the ***characteristic function*** $\chi_{A'}: A \to \{0,1\}$ is defined as follows:

$$\chi_{A'}(a) = \begin{cases} 1, & a \in A' \\ 0, & a \in A - A' \end{cases}$$

- **Example：let** ***A***=**{*a*,*b*,*c*}**, Each subset ***A'*** of ***A*** corresponds to a **characteristic function**, and different subsets correspond to different characteristic functions. **Such as :**

$$\chi_{\varnothing} = \{<a,0>,<b,0>,<c,0>\}$$

$$\chi_{\{a,b\}} = \{<a,1>,<b,1>,<c,0>\}$$

- The characteristic function of a set is a **detector** that determines whether an element belongs to the set; it is also known as an **indicator function** that represents whether an event occurs.

(5) Let ***R*** be an equivalence relation on ***A*** . Define

$$\mathbf{g} : A \to A/R$$

$$\mathbf{g}(a) = [a],\ \forall a \in A$$

Then **g** is called the ***natural mapping*** **(function)** from ***A*** to the quotient set ***A/R***.